Lecture 10, E6712

Two Probability Distributions

Rayleigh Distribution

Consider the noise process $n(t)$

$$\begin{aligned} n(t) =& r(t)e^{j\phi(t)} \\ =& x(t) + jy(t) \end{aligned}$$

where $r(t)$ is the magnitude or envelope and $\phi(t)$ is the phase, $x(t)$ is the in-phase, and $y(t)$ is the quadrature component.

If both random processes $x(t)$ and $y(t)$ are statistically independent Gaussian distributed with the same variance and zero mean, then their joint probability density function is

$$\begin{aligned} P(x,y) =& P(x)P(y) \\ =& \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \end{aligned}$$

Transform differential areas by using $dxdy = rdrd\phi$ gives the joint probability density function of $r(t)$ and $\phi(t)$ as

$$P(r,\phi) = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \qquad r \in [0,\infty),\ \phi \in [-\pi,\pi],$$

We get

$$\begin{aligned} P(r) =& \int_{-\pi}^{\pi} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} d\phi = \boxed{\frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}} \leftarrow \textit{Rayleigh distribution} \\ P(\phi) =& \int_{0}^{\infty} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = \boxed{\frac{1}{2\pi}} \leftarrow \textit{uniform distribution} \end{aligned}$$

Note that the random variables r and ϕ are statistically independent:

$$P(r,\phi) = P(r)P(\phi).$$

Rice Distribution

Consider the noise process $n(t)$

$$\begin{aligned} n(t) =& r(t)e^{j\phi(t)} \\ =& x(t) + jy(t) \end{aligned}$$

where $r(t)$ is the magnitude or envelope and $\phi(t)$ is the phase, $x(t)$ is the in-phase, and $y(t)$ is the quadrature component.

If both random processes $x(t)$ and $y(t)$ are statistically independent Gaussian distributed with the same variance σ^2 and means μ_x and μ_y, then probability density function of r is the Rician probability density function given by

$$p(r) = \frac{r}{\sigma^2} e^{-\frac{(r^2+m^2)}{2\sigma^2}} I_0(rm/\sigma^2),$$

where $m^2 = \mu_x^2 + \mu_y^2$ and I_0 is the modified 0-th-order Bessel function of the first kind given by

$$I_0(x) \triangleq \frac{1}{\pi} \int_0^{\pi} \exp(x \cos\theta) d\theta.$$

Note that m and σ^2 are not the mean value and standard deviation for the Rician noise.

Characterization of Fading Channels The mobile wireless channel varies over time and over frequency.

Large-scale fading refers to the channel variations due to path loss with the increase in distance and to shadowing by large objects such as buildings and hills. This occurs as the mobile moves through a distance of the order of the cell size, and is typically frequency independent.

Small-scale fading refers to the channel variations due to the constructive and destructive interference of the multiple signal propagation paths between the transmitter and receiver. This occurs at the spatial scale of the order of the carrier wavelength, and is frequency dependent.

We will consider the multipath fading.

The received signal is a superposition of of a number of signals arriving from different paths.

Let the transmitted bandpass signal be

$$s(t) = \Re\{u(t)e^{j2\pi f_c t}\}$$

Assume that there is a discrete number of propagation paths, each characterized by some random delay $\tau_n(t)$ and some attenuation $\alpha_n(t)$.

Then the received signal is

$$\begin{aligned} x(t) =& \sum_n \alpha_n(t)s(t-\tau_n(t)) \\ =& \Re\Big\{\sum_n \alpha_n(t)u(t-\tau_n(t))e^{j2\pi f_c(t-\tau_n(t))}\Big\} \\ =& \Re\Big\{\underbrace{\Big[\sum_n \alpha_n(t)e^{j2\pi f_c \tau_n(t)}u(t-\tau_n(t))\Big]}_{r(t)} e^{j2\pi f_c t}\Big\} \end{aligned}$$

Consequently, the received baseband equivalent signal is

$$r(t) = \sum_n \alpha_n(t)e^{j2\pi f_c \tau_n(t)}u(t-\tau_n(t))$$

This is equivalent to a passage through a time variant channel with impulse response

$$c(\tau;t) = \sum_n \alpha_n(t)e^{-j2\pi f_c \tau_n(t)}\delta(\tau-\tau_n(t))$$

We have

$$r(t) = \int_{-\infty}^{\infty} c(\tau;t)u(t-\tau)d\tau$$

For a continuum of paths, the received bandpass signal is

$$x(t) = \int_{-\infty}^{\infty} \alpha(\tau;t)s(t-\tau)d\tau$$

where $\alpha(\tau;t)$ denotes the attenuation of signal components at delay τ and at time instant t.

We have

$$x(t) = \Re\Big\{\int_{-\infty}^{\infty} \alpha(\tau;t)u(t-\tau)e^{j2\pi f_c(t-\tau)}\Big\}$$

The equivalent low-pass time-variant impulse response is

$$c(\tau;t) = \alpha(\tau;t)e^{-j2\pi f_c \tau}$$

Consider the case of sending an unmodulated carrier of frequency f_c through a channel with discrete multipath components.

The received bandpass signal

$$r(t) = \sum_n \alpha_n(t) e^{j2\pi f_c \tau_n(t)} = \sum_n \alpha_n(t) e^{\theta_n(\tau)}$$

$\alpha_n(t)$ changes slowly, $\theta_n(\tau)$ changes fast.

If there is a large number of multipath components, then we can invoke the central limit theorem to conclude that the impulse response $c(\tau; t)$ is a complex Gaussian process in t.

Rayleigh Fading: When $c(\tau; t)$ is modelled as a complex *zero-mean* Gaussian process, then its envelope $|c(\tau; t)|$ is Rayleigh distributed.

Ricean Fading: When $c(\tau; t)$ is modelled as a complex *non zero-mean* Gaussian process, *i.e.,* when there is a fixed direct propagation path between the transmitter and the receiver.

The autocorrelation function for a zero-mean WSS process $c(\tau; t)$ is

$$\varphi_c(\tau, \tau'; \Delta t) = \frac{1}{2} E\{c^*(\tau; t) c(\tau'; t + \Delta t)\} = \frac{1}{2} \varphi_c(\tau; \Delta t) \delta(\tau - \tau')$$

For the last equality, we assume that the attenuation and the phase shift of the channel associated with the path delay τ is uncorrelated with the attenuation and the phase shift of the channel associated with the path delay τ'

For $\Delta t = 0$, $\varphi_c(\tau_1; 0) = \varphi_c(\tau)$ is known as the *multipath intensity profile of the channel.*

T_m is the *multipath spread of the channel.* It determines the delay up to which the signals are still considered to be correlated.

$\varphi_c(\tau)$ can be measured by sending a very narrow pulse through the channel and correlating the received signal with a delayed version of itself.

Another way to characterize the time variant multipath channel is

$$C(f; t) = \int_{-\infty}^{\infty} c(\tau; t) e^{-j2\pi f \tau} d\tau$$

and

$$\begin{aligned}
\phi_c(f, f'; \Delta t) =& \frac{1}{2} E\{C^*(f; t) C(f'; t + \Delta t)\} \\
=& \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\{c^*(\tau; t) c(\tau'; t + \Delta t)\} e^{j2\pi(f\tau - f'\tau')} d\tau d\tau' \\
=& \frac{1}{2} \int_{-\infty}^{\infty} \varphi_c(\tau; \Delta t) e^{j2\pi(f - f')} d\tau \\
=& \varphi_c(\Delta f, \Delta t)
\end{aligned}$$

For $\Delta t = 0$,

$(\Delta f)_c \approxeq 1/T_m$ is the coherence bandwidth of the channel.

When $(\Delta f)_c$ is small compared to the bandwidth of the of the signal, then the channel is *frequency selective.* Otherwise is frequency nonselective.

Time variations of the channel, *i.e.,* change of $\phi_c(\Delta f; \Delta t)$ as a function of Δt.

We define

$$S_c(\Delta f; \lambda) = \int_{-\infty}^{\infty} \phi_c(\Delta f; \Delta t) e^{-j2\pi\lambda\Delta t} d\Delta t$$

When $\Delta f = 0$, we have the *Doppler power spectrum*

$$S_c(\lambda) = \int_{-\infty}^{\infty} \phi_c(\Delta t) e^{-j2\pi\lambda\Delta t} d\Delta t$$

The range of values of λ over which $S(\lambda)$ is non-zero is B_d, the *Doppler spread of the channel.*

The *coherence time*:

$$(\Delta t)_c \approxeq \frac{1}{B_d}$$

Large coherence time means small channel time variations.

"Good" channels have large $(\Delta t)_c$ and large $(\Delta f)_c$.

Scattering function:

$$\begin{aligned} S(\tau; \lambda) &= \int_{-\infty}^{\infty} \phi_c(\tau; \Delta t) e^{-j2\pi\lambda\Delta t} d\Delta t \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_c(\Delta f; \Delta t) e^{j2\pi\tau\Delta f} e^{-j2\pi\lambda\Delta t} d\Delta f d\Delta t \\ &= \int_{-\infty}^{\infty} S_c(\Delta f; \lambda) e^{-j2\pi\tau\Delta f} d\Delta f \end{aligned}$$

The Effect of Signal Properties on the Choice of Channel Model

Let $u(t)$ be the baseband signal and $U(f)$ its Fourier transform

Ignoring additive noise, we have

$$\begin{aligned} r(t) &= \int_{-\infty}^{\infty} c(\tau; t) u(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} C(f; t) U(f) e^{-j2\pi f t} dt \end{aligned}$$

If $u(t)$ has a bandwidth W such that $W > (\Delta f)_c$, the channel is frequency selective.

If

$$W \approxeq \frac{1}{T} \ll \frac{1}{T_m} \approxeq (\Delta f)_c$$

the channel is frequency-nonselective. In other words, all the all the frequency components of $U(f)$ undergo the same attenuation and phase shift in transmission through the channel.

$C(f; t)$ is approximately constant over W:

$$r(t) = C(0; t) u(t)$$

We denote $C(0; t) = \alpha(t) e^{-j\phi(t)}$. On a frequency-nonselective noisy channel, the received signal is given by

$$r(t) = \alpha(t) e^{-j\phi(t)} u(t) + z(t)$$

Binary Signaling Over Frequency-Nonselective Channels

On a frequency-nonselective channel, the received signal is given by

$$r(t) = \alpha(t)e^{-j\phi(t)}g(t) + z(t)$$

Assume slow fading $\therefore$ $\alpha(t)$ is constant and $\phi(t)$ is constant and can be estimated $\therefore$ *coherent* detection is possible for

$$r(t) = \alpha e^{-j\phi}g(t) + z(t)$$

Do we need to know α?

Error probability performance for BPSK:

$$P_e(\alpha) = Q\left(\sqrt{\frac{2\alpha^2 E_b}{\mathcal{N}_0}}\right) = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{\alpha^2 E_b}{\mathcal{N}_0}}\right)$$

$$\therefore$$

$$P_e(\gamma_b) = Q(\sqrt{2\gamma_b}) \quad \text{where} \quad \gamma_b = \frac{\alpha^2 E_b}{\mathcal{N}_0}$$

$$P_e = \int P_e(\gamma_b)p(\gamma_b)d\gamma_b$$

The distribution of α is Rayleigh; the distribution of α^2 and thus γ_b is chi-square with two degrees of freedom:

$$p(\gamma_b) = \frac{1}{\overline{\gamma}_b}e^{-\gamma_b/\overline{\gamma}_b},$$

where

$$\overline{\gamma}_b = \frac{E_b}{\mathcal{N}_0}E(\alpha^2)$$

We have

$$P_e = \int P_e(\gamma_b)p(\gamma_b)d\gamma_b = \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}_b}{1+\overline{\gamma}_b}}\right)$$

On channels for which the fading is too rapid for phase estimation, we can use DPSK which requires phase stability over only two consecutive signaling intervals.

Error probability performance for DPSK:

$$P_e(\gamma_b) = \frac{1}{2}e^{-\gamma_b}$$

$$P_e = \int P_e(\gamma_b)p(\gamma_b)d\gamma_b$$

$$\therefore$$

$$P_e = \frac{1}{2(1+\overline{\gamma}_b)}$$

Other fading models.